

In [PM], Russell & Whitehead show the reduction of these into *logic* (81):

- 1.1. Set Theory
- 1.2. Arithmetic
- 1.3. Algebra
- 1.4. Abstract Algebra (via the *logic of relations*)
- 1.5. Analysis
- 1.6. Analytic Geometry (via 3 & 5)

▷ The *logic of PM* to which all these are reducible is a superlogic of the Aristotelian (Assertoric) *Syllogistic* (81).¹

▷ The logic of PM consists of three parts (81):

- 2.1. Logic of *Propositions*
- 2.2. Logic of *Classes*
- 2.3. Logic of *Relations*

Remark: The development of the logics of propositions (2.1) & classes (2.2) can be traced back to Aristotle, Leibniz, Boole and others that came after them. The logic of relations (2.3) can perhaps be traced back to De Morgan, Peirce, Schröder, and others. Frege did his own thing with all three, but he was in no way the originator of either.

▷ The logic of PM can be obtained from three primitive notions (81):

- 3.1. the relation of *set membership* (i.e., '∈')
- 3.2. the *Sheffer stroke* (i.e., '|')
- 3.3. *universal quantification* (i.e., '(∀x)' for a variable x)

Remark: It turns out that (3.1-3) are not minimal. Moses Schönfinkel in "On the Building Blocks of Mathematical Logic" has managed to extend the Sheffer

¹Aristotle's Syllogistic can be thought of as a restricted *logic of classes*, where the only *relation* between classes that is considered is the relation of class inclusion (i.e., '⊂'). It has also been equated with monadic predicate calculus, that is, the 1-variable fragment of FOL.

stroke to what we may call *generalized Sheffer stroke* (i.e., '|^x' for a variable x). Using this generalized Sheffer stroke, we can derive universally-quantified formulas as follows (Schönfinkel 1924, §1):

- | | | | |
|-----|------------------------------|-----|--|
| A1. | $\neg p$ | for | $p ^x p$. |
| A2. | $p \vee q$ | for | $(p ^y p) ^x (q ^y q)$. |
| A3. | $(\forall \alpha) f(\alpha)$ | for | $(f(\alpha) ^y f(\alpha)) ^\alpha (f(\alpha) ^y f(\alpha))$. |

Thus, if Quine is right about (3.1-3), then the logic of PM, and therefore the entirety of mathematics (!) can be obtained from two primitive notions:

- 3.1. the relation of *set membership* (i.e., '∈')
- 3.2*. the *generalized Sheffer stroke* (i.e., '|^x' for a variable x)

▷ Quine's logic of New Foundations (NF) consists of the following definitions:

- | | | | |
|------|-----------------------------------|-----|---|
| D1. | $\neg \phi$ | for | $\phi \phi$ |
| D2. | $(\phi \wedge \psi)$ | for | $\neg(\phi \psi)$ |
| D3. | $(\phi \rightarrow \psi)$ | for | $(\phi \neg \psi)$ |
| D4. | $(\phi \vee \psi)$ | for | $(\neg \phi \rightarrow \psi)$ |
| D5. | $(\phi \equiv \psi)$ | for | $((\phi \psi) (\phi \vee \psi))$ |
| D6. | $(\exists \alpha) \phi$ | for | $\neg(\forall \alpha) \neg \phi$ |
| D7. | $(\alpha \subset \beta)$ | for | $(\forall \gamma)((\gamma \in \alpha) \rightarrow (\gamma \in \beta))$ |
| D8. | $(\alpha = \beta)$ | for | $(\forall \gamma)((\alpha \in \gamma) \rightarrow (\beta \in \gamma))$ |
| D9. | $((\iota \alpha) \phi \in \beta)$ | for | $(\exists \gamma)((\gamma \in \beta) \wedge (\forall \alpha)((\alpha = \gamma) \equiv \phi))$ |
| D10. | $(\beta \in (\iota \alpha) \phi)$ | for | $(\exists \gamma)((\beta \in \gamma) \wedge (\forall \alpha)((\alpha = \gamma) \equiv \phi))$ |
| D11. | $\hat{\alpha} \phi$ | for | $(\iota \beta)(\forall \alpha)((\alpha \in \beta) \equiv \phi)$ |
| D12. | $\{\alpha\}$ | for | $\hat{\beta}(\beta = \alpha)$ |
| D13. | $\{\alpha, \beta\}$ | for | $\hat{\gamma}((\gamma = \alpha) \vee (\gamma = \beta))$ |
| D14. | $\langle \alpha, \beta \rangle$ | for | $\{\{\alpha\}, \{\alpha, \beta\}\}$ |
| D15. | $\hat{\alpha} \hat{\beta} \phi$ | for | $\hat{\gamma}(\exists \alpha)(\exists \beta)((\gamma = \langle \alpha, \beta \rangle) \wedge \phi)$ |

The theorems of NF, Quine claims, are all derivable from a single postulate (P1, p. 89) by means of inference rules (R1-5).